RESEARCH STATEMENT

WHITNEY BERARD

My research is in the area of algebraic number theory, and is primarily concerned with Galois representations. The main object of algebraic number theory is to figure out under what conditions polynomial equations have solutions in a given field. Galois theory was invented and developed over the past two centuries for precisely this purpose. In particular, the Galois group of a polynomial is a finite group that describes the interactions of the roots of that polynomial. Galois representations give us information about a Galois group by mapping it into a relatively simple group like the group of two by two matrices over a field.

It is often useful to translate from this algebraic setting into an analytic setting, where we have a different set of tools we can use. In the context of Galois theory, a conjecture of Serre provides us with a connection between between algebraic objects given by 2-dimensional Galois representations and analytic objects given by certain holomorphic functions on the complex plane, called modular forms.

My work is concerned with generalizations of Serre's conjecture that predict which Serre weights (a representation-theoretic avatar for the weight of a classical modular form) are possible for a given Galois representation. I was able to make progress in this direction by providing additional computational evidence and showing some internal consistency among these generalized conjectures. In particular, I will show in my dissertation that in dimension four, the set of explicit Serre weights is closed under taking tensor products.

In this statement, I first give a brief overview of the background and motivation for the problem I am working on, then in Section 2 I describe the main question that drives my research. I go into more detail on the objects and tools I work with in Section 3, and describe the result I will show in my dissertation in Section 4. Notably, there is an important subset of the objects I work with — weights — that can be more easily understood graphically. In Section 5 of this statement I describe how I have used 3D pictures, viewable with red-cyan 3D glasses, in order to think about this set of weights. I also mention how this 3D picture relates to a particular result that will appear in my dissertation.

1. Background and Motivation

1.1. **Notation and Definitions.** Let p be a prime number. We define an absolute value $|\cdot|_p$ on $\mathbb{Q}$, called the **p-adic absolute value** in the following way: For any $\frac{m}{n} \in \mathbb{Q}^\times$, we may write $\frac{m}{n} = p^r \frac{m'}{n'}$ for some $r \in \mathbb{Z}$ and $m', n' \in \mathbb{Z}$ not divisible by p. Define $\left|\frac{m}{n}\right|_p = \frac{1}{p^r}$. This absolute value provides us with a metric on $\mathbb{Q}$, and the completion of $\mathbb{Q}$ with respect to this metric is the **field $\mathbb{Q}_p$ of p-adic numbers**.

The objects that I study are called Galois representations. If K is a field, and $\overline{K}$ is an algebraic closure of K, then the **absolute Galois group** of K is the group of automorphisms of $\overline{K}$ which fix K. We use the notation $G_K = \mathrm{Gal}(\overline{K}/K)$ for the absolute Galois group of a field K. **Galois representations** are representations (in the usual representation theory sense) of an absolute Galois group.

Of particular interest are representations of $G_{\mathbb{Q}}$. Fix an odd prime p for the remainder of this statement. We will consider these Galois representations with p-adic coefficients (that is, representations into $\mathrm{GL}_n(\overline{\mathbb{Q}}_p)$), since representations of this type arise naturally from algebraic geometry.

1.2. **Serre's Conjecture and Generalizations.** A conjecture of J.P. Serre [Ser87] gives us a relationship between certain Galois representations and modular forms, and provides specific information about the modular forms that are attached to a given Galois representation. Formally speaking, Serre conjectured that every odd, irreducible, continuous representation $\overline{r} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ arises from a modular eigenform. He also gave a recipe for the minimal weight and level of a modular eigenform giving rise to $\overline{r}$.

This conjecture was proved in 2008 by Khare and Wintenberger [KW09]. Much recent work has been done to generalize this conjecture by looking at fields other than $\mathbb{Q}$, and dimensions greater than two. For example, the conjectures of [GHS15] study representations of the form $\overline{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, where F is a number field (a finite extension of $\mathbb{Q}$) and $n \geq 2$.

2. The main question

In order to study these Galois representations, one often restricts to the "local" picture in the following way. Let ℓ be a prime number. Then $G_{\mathbb{Q}_\ell} \subset G_{\mathbb{Q}}$, and the restriction of a representation of $G_{\mathbb{Q}}$ to $G_{\mathbb{Q}_\ell}$ provides us with important invariants. The most subtle of these is when $\ell = p$, and I proceed by studying representations of $G_{\mathbb{Q}_p}$. Recall that p is an odd prime, fixed for the remainder of this statement.

My research is concerned with an important subclass of representations of $G_{\mathbb{Q}_p}$ called **crystalline representations**. Roughly speaking, crystalline representations are defined by some algebraic conditions motivated by geometry. If $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ is a crystalline representation, then associated to ρ are two invariants:

- an n-tuple of integers called **Hodge-Tate weights**, and
- a semisimple $\mathrm{mod}\, p$ representation $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$.

There may be many different crystalline representations ρ which all reduce to the same $\overline{\rho}$. We call these the **lifts** of $\overline{\rho}$. If a lift ρ has distinct Hodge-Tate weights $x_1 > \cdots > x_n$, we say that ρ has **Hodge type** $(x_1 - (n-1), x_2 - (n-2) \ldots, x_{n-1} - 1, x_n)$.

In this context, a Serre weight is an irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_n(\mathbb{F}_p)$. By a theorem of Steinberg, a Serre weight can be represented by $F(\mathbf{a})$, where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $0 \leq a_i - a_{i+1} \leq p-1$, and $F(\mathbf{a}) \cong F(\mathbf{a}')$ if $\mathbf{a} - \mathbf{a}' \in (p-1, \ldots, p-1)\mathbb{Z}$.

Given a semisimple $\mathrm{mod}\, p$ representation $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, define the **crystalline weights for** $\overline{\rho}$, $W^{\exists}_{\mathrm{cris}}(\overline{\rho})$, to be the set of Serre weights $\mathbf{a}$ such that $\overline{\rho}$ has a crystalline lift that is regular of Hodge type $\mathbf{a}$. If $F(\mathbf{a}) = F(\mathbf{a}')$ then $\overline{\rho}$ has a lift of Hodge type $\mathbf{a}$ if and only if it has one of type $\mathbf{a}'$, so the set $W^{\exists}_{\text{cris}}(\overline{\rho})$ is well-defined.

Now armed with the set $W^{\exists}_{\text{cris}}(\overline{\rho})$, we ask the following question:

Question 1. *Given a* $\bmod p$ *representation* $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$*, what is* $W^{\exists}_{\text{cris}}(\overline{\rho})$*?*

3. Some subsets of $W^{\exists}_{\text{cris}}(\overline{\rho})$

Our strategy to study Question 1 is to construct some subsets of Serre weights which we expect are contained in $W^{\exists}_{\text{cris}}(\overline{\rho})$. In particular, we build up the following subsets:

(1) $W_{\text{obv}}(\overline{\rho})$: The set of *obvious weights*.

(2) $\mathcal{C}(W_{\text{obv}}(\overline{\rho}))$: Here $\mathcal{C}$ is a certain "closure" operator. This provides additional *shadow weights*.

(3) $W_{\text{expl}}(\overline{\rho})$: The set of *explicit predicted weights*. This provides additional *obscure weights*.

3.1. Obvious Weights.

If $\overline{\rho}|_{I_{\mathbb{Q}_p}}$ is semisimple, then there is a certain class of lifts of $\overline{\rho}|_{I_{\mathbb{Q}_p}}$ that are easy to write down. For any n-tuple $\{a_i\}$, there exists a crystalline representation $\Psi_{\{a_i\}} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{Z}}_p)$ with Hodge-Tate weights $\{a_i\}$ and

$$\overline{\Psi}_{a_i}|_{I_{\mathbb{Q}_p}} = \omega_n^m \oplus \omega_n^{pm} \oplus \cdots \oplus \omega_n^{p^{n-1}m},$$

where ω_n is the character of order $p^n - 1$, and $m = a_{n-1}p^{n-1} + \cdots + a_1 p + a_0$.

Given a $\bmod p$ representation $\overline{\rho}$ such that $\overline{\rho}|_{I_{\mathbb{Q}_p}}$ is semisimple, we say that $\bigoplus_j \Psi_{\{a_{ij}\}_i}$ is an **obvious lift** of $\overline{\rho}$ if $\bigoplus_j \overline{\Psi}_{\{a_{ij}\}_i}|_{I_{\mathbb{Q}_p}} = \overline{\rho}|_{I_{\mathbb{Q}_p}}$. For example, suppose $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ has $\overline{\rho}|_{I_{\mathbb{Q}_p}} \cong \omega^a \oplus \omega^b \oplus \omega^c$ with $a > b > c$. Then $\Psi = \Psi_{\{b+p-1\}} \oplus \Psi_{\{a\}} \oplus \Psi_{\{c\}}$ is an obvious lift of $\overline{\rho}$.

We define the **set of obvious weights**, $W_{\text{obv}}(\overline{\rho})$, to be the set of Serre weights $\mathbf{a}$ such that $\overline{\rho}$ has an obvious lift of Hodge type $\mathbf{a}$.

3.2. Shadow Weights.

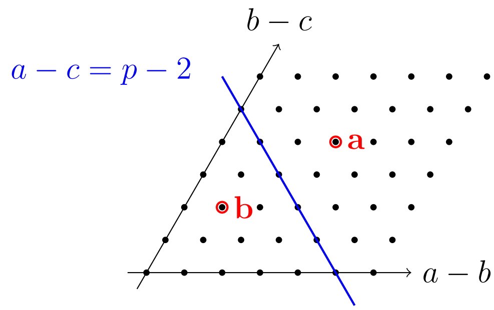
Using the obvious weights as a starting point, we perform some "reflections" to obtain more weights. The Breuil-Mézard Conjecture is an important ingredient in these investigations. It is an immediate consequence of this conjecture that if $a \in W_{\text{cris}}(\overline{\rho})$ and $F(a)$ is a Jordan-Hölder factor of they Weyl module $W(b)$, then $b \in W_{\text{cris}}(\overline{\rho})$.

We define $\mathcal{C}(W_{\text{obv}}(\overline{\rho}))$ to be the smallest set satisfying the following condition: If $a \in W_{\text{cris}}(\overline{\rho})$ and $F(a)$ is a Jordan-Hölder factor of the Weyl module $W(b)$, then $b \in W_{\text{cris}}(\overline{\rho})$. We call the weights in $\mathcal{C}(W_{\text{obv}}(\overline{\rho})) \setminus W_{\text{obv}}(\overline{\rho})$ **shadow weights**.

To find shadow weights, we want to find the Jordan-Hölder factors $F(\mathbf{b}_i)$ of Weyl groups $W(\mathbf{a})$. Given an n-tuple $\mathbf{a}$, these Jordan-Hölder factors are obtained by reflecting $\mathbf{a}$ about certain hyperplanes in a weight space, and then imposing additional conditions on the reflections that are allowed to survive. Namely, the $\mathbf{b}_i$ need to lie "below" $\mathbf{a}$ in some sense.

In the dimension three case, the picture goes like this: Start with a Serre weight $F(\mathbf{a}) = F(a, b, c)$. We view this weight as the point $(a - b, b - c)$ in two dimensional space. If

$a-c>p-2$, then the Jordan-Hölder factors of $W(\mathbf{a})$ are $F(\mathbf{a})$ and $F(\mathbf{b})$, where $\mathbf{b}$ is the reflection of $\mathbf{a}$ about the hyperplane $a-c=p-2$.



In the dimension four case, the appropriate picture is a cube with side length p, and there are four (hyper)planes that slice this 3D cube into six "alcoves". The only way I have found to effectively understand this picture is with a 3D picture viewable by 3D glasses. This 3D picture and an accompanying example are available in Section 5.

3.3. **Obscure Weights.** When $n>3$, there are some additional weights lying on the boundaries of the alcoves that are not picked up by either the obvious or shadow weights. Again, let $\overline{\rho}$ be a mod p representation such that $\overline{\rho}$ is semisimple. Then we can construct a class of weights $W_{\text{expl}}(\overline{\rho})$ called *explicit weights*. This includes both obvious and shadow weights. And when $n\geq 3$, there are additional *obscure weights* in this set.

There is an explicit recipe for finding the set $W_{\text{expl}}(\overline{\rho})$, though it is a bit technical. Here is one example of constructing explicit weights in dimension 4:

Suppose $\overline{\rho}=\omega^a\oplus\omega^b\oplus\omega^c\oplus\omega^d$, with $a>b>c>d$. We write $\overline{\rho}$ as a decomposition: $\overline{\rho}=\overline{\rho}^{(1)}\oplus\overline{\rho}^{(2)}$, where

$$\overline{\rho}^{(1)}=\omega^a\oplus\omega^c,\qquad \overline{\rho}^{(2)}=\omega^b\oplus\omega^d.$$

A Serre weight $\mathbf{a}=(x-3,y-2,z-1,w)$ is an explicit weight of $\overline{\rho}$ if either:

- $W_{\text{expl}}(\overline{\rho}^{(1)})\cap JH(W(x-1,z))\neq\varnothing$ and $W_{\text{expl}}(\overline{\rho}^{(2)})\cap JH(W(y-1,w))\neq\varnothing$, or
- $W_{\text{expl}}(\overline{\rho}^{(1)})\cap JH(W(x-2,w))\neq\varnothing$ and $W_{\text{expl}}(\overline{\rho}^{(2)})\cap JH(W(y-1,z))\neq\varnothing$.

4. An expected result

In dimension four and larger, there exist mod p representations $\overline{\rho}$ such that $\mathcal{C}(W_{\text{obv}}(\overline{\rho}))\subsetneq W_{\text{expl}}(\overline{\rho})\subset W^{\exists}_{\text{cris}}(\overline{\rho})$. Of course, it would be nice if we knew that $W_{\text{expl}}(\overline{\rho})=W^{\exists}_{\text{cris}}(\overline{\rho})$, but so far there is not much evidence that this must be the case except in dimensions two and three. To try and get a handle on this in larger dimensions, we ask the question:

Question 2. *Is $W_{\text{expl}}(\overline{\rho})$ closed under taking tensor products and inductions?*

In dimension four, the tensor product part of this question can by posed in the following way. Suppose $\overline{\rho}'$ has a lift ρ' with Hodge-Tate weights $(a,0)$, and $\overline{\rho}''$ has a lift ρ'' with Hodge-Tate weights $(b,0)$ such that $1\leq a\leq p$ and $1\leq b-a\leq p$. Then $\overline{\rho}=\overline{\rho}'\otimes\overline{\rho}''$ has $\rho=\rho'\otimes\rho''$

as a lift. We know that ρ has Hodge-Tate weights $(a+b, b, a, 0)$. Is the corresponding Serre weight $(a+b-3, b-2, a-1, 0)$ in $W_{\text{expl}}(\overline{\rho})$?

In my dissertation, I expect to show that the answer is yes. Moreover, $\mathcal{C}(W_{\text{obv}}(\overline{\rho}))$ is not enough in general. That is, there exist choices for a, b such that $(a+b-3, b-2, a-1, 0)$ is an *obscure weight*.

5. Visualizing Shadow Weights

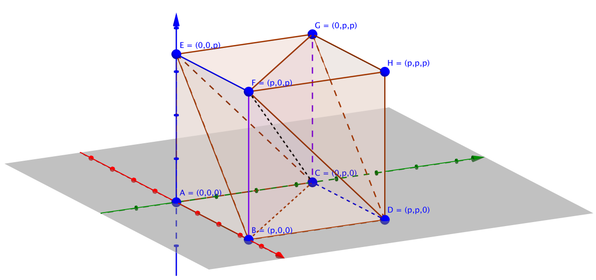
Recall that the *shadow weights* we study are found by looking at the list of Jordan-Hölder factors of the Weyl modules $W(\mathbf{a})$. For instance, in dimension four one of these decompositions looks like:

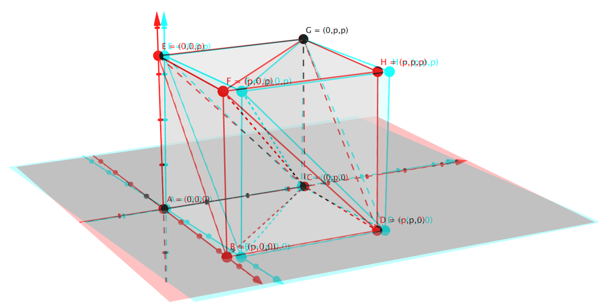
$$W(\mathbf{a}) = F(\mathbf{a}) + F(\mathbf{b}_3) + F(\mathbf{b}_2) + F(\mathbf{b}_1) + F(\mathbf{b}_0),$$

where all the $\mathbf{b}_i$'s are successive reflections of $\mathbf{a}$.

For four-dimensional representations, we visualize a weight $F(a-3, b-2, c-1, d)$ as the point $(a-b, b-c, c-d)$ in 3D space. The weight space we are concerned with can be visualized as a cube with side length p, with four planes cutting the cube into six pieces, called alcoves. I found all this much easier to work with once I was able to visualize the cube properly.

On the left is a picture of the dimension four alcoves, and on the right is a similar picture rendered to be seen with 3D red-cyan glasses. (If you email me with a mailing address I will happily send you a pair.)



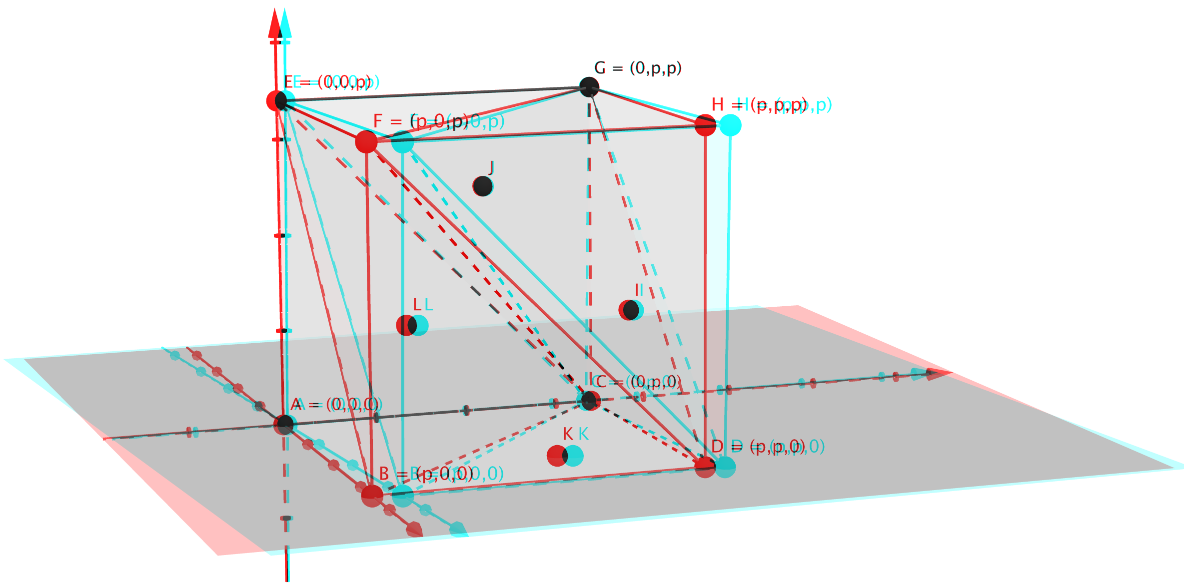


For me, the interesting cases to look at are when a weight lies on one of the boundaries. For example, let's look at a (Hodge-Tate) weight $\lambda = (a, b, c, d)$ with $(a-b, b-c, c-d) = (x, p, x)$, where $x < p/2$. To find the Jordan-Hölder factors of $W(a-3, b-2, c-1, d)$, we look at all the reflections of λ about the hyperplanes inside this cube. When λ is on a boundary, it happens that the only Jordan-Hölder factors that survive are those that end up on an "upper" boundary of an alcove.

In the following picture, we take the point $I = (x, p, x)$, and reflect it about the various hyperplanes to obtain J, K, L. If you've got your trusty 3D glasses on, you'll notice that:

- $I = (x, p, x)$ is on an upper boundary of the alcove $FGCD$,
- $J = (0, p-x, 2x)$ is on a lower boundary of the alcove $EFGC$,

- $K = (2x, p-a, 0)$ is on a lower boundary of the alcove $BCDF$, and
- $L = (a, p-2a, a)$ is on the upper boundary of the alcove $ABCE$.



Since J, K are not on upper boundaries of any alcove, those Serre weights do not survive in the Jordan-Hölder decomposition. The Serre weight corresponding to the point I will be a shadow weight if the Serre weight corresponding to L is already known to be in $W^{\exists}_{\text{cris}}(\overline{\rho})$.

In my dissertation, I show that for certain four-dimensional representations $\overline{\rho}$ arising from the tensor product of two two-dimensional representations, the Serre weight corresponding to I in the picture above is a shadow weight. This happens, specifically, because the Serre weight corresponding to L in the picture is an obvious weight for the representations $\overline{\rho}$.

References

[GHS15] Toby Gee, Florian Herzig, and David Savitt, *General serre weight conjectures.*

[Her06] Florian Herzig, *The weight in a Serre-type conjecture for tame n-dimensional Galois representations*, Phd thesis, Harvard University, May 2006.

[KW09] Chandrashekhar Khare and Jean-Pierre Wintenberger, *Serres modularity conjecture (i)*, Inventiones Mathematicae **178** (2009), 485504.

[Ser87] Jean-Pierre Serre, *Sur les représentations modulaires de degré 2 de* $\text{Gal}(\overline{\mathbf{q}}/\mathbf{Q})$, Duke Math. J. **54** (1987), no. 1, 179–230.